

Flocking particles in a non-Newtonian shear thickening fluid

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Abstract

We prove existence and uniqueness of strong solutions to the Cucker–Smale flocking model coupled with an incompressible viscous non-Newtonian fluid, with the stress tensor of a power-law structure for $p \geq \frac{11}{5}$. The coupling is performed through a drag force on a periodic spatial domain \mathbb{T}^3 .

1 Introduction

Mathematical models of self-propelled agents with non-local interactions provide a way to describe a wide range of phenomena in natural sciences: physics, biology, but also in economics or even in robotics. The literature concentrates on analysis of time asymptotics [23] pattern formation [22, 33] and study of models with forces that simulate various natural factors ([7, 16] in the deterministic case) or ([11] in the stochastic one). The other variations of the model include forcing particles to avoid collisions [9] or to aggregate under the leadership of certain individuals [10].

We concentrate on the Cucker–Smale (CS) flocking model describing a collective self-driven motion of self-propelled particles with a tendency to flock. The system has been introduced by Cucker and Smale in [12] in 2007 and it initiated intensive study of the subject from the mathematical point of view. The vast literature on CS concerns mostly qualitative analysis [6, 33, 21]. Simple form of the system allows unexpectedly to find answers for questions concerning the structure of solutions like aggregation with leaders [10, 32], collision avoidance (XX), cluster formation (XX), Mosch-Tadmor (XX). The theory contains also examination of systems with stochastic forces [?, 16, 24], with singular weight

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ψ [25, 1, 29, 30] and on the passage from the particle the kinetic description for the CS system [25, 26, 28] and general models [5, 13, 14].

The present paper considers one of the other directions of research. Our subject is motion of agents described by the kinetic CS equation

$$\partial_t f + v \cdot \nabla_x f + \operatorname{div}_v(F(f)f) = 0, \quad (1.1)$$

submerged in a non-Newtonian viscous incompressible fluid. In parallel to the analysis of the kinetic models themselves, research in coupling models of kinetic theory with models of hydrodynamics was investigated in (see [4, 19, 20]) and they are a part of large theory called complex flows. Our motivation comes from results for the complex flow models, here we shall mentioned about [18, 8] concerning the Fokker–Planck equation coupled with the Navier–Stokes system. We aim at proving well posedness of the coupled system guaranteeing global in time solvability for arbitrary large data with unique regular solutions. Note that for the classical Navier–Stokes equations we are still not able to consider general smooth solutions, hence application of the non-Newtonian concept of description of flow allows to obtain stronger results than for the Newtonian case like in [2] or [4] for the pure Vlasov case.

Our goal is to consider particles embedded in an incompressible viscous non-Newtonian shear thickening fluid, i.e. we aim to couple (1.1) with the system

$$\begin{cases} \partial_t u + (u \cdot \nabla_x)u + \nabla_x \pi - \operatorname{div}_x(\tau) &= G_{ext}, \\ \operatorname{div}_x u &= 0, \end{cases} \quad (1.2)$$

which describes the motion of such a fluid. We will consider the three-dimensional flows. The function

$$u = u(t, x) = (u_1(t, x), u_2(t, x), u_3(t, x))$$

represents velocity of the fluid at the position x and time t . Equation (1.2)₂ expresses the conservation of mass (as well as the incompressibility constraint), while (1.2)₁ expresses the conservation of momentum. The term τ in (1.2)₁ denotes a symmetric stress tensor that depends on Du — the symmetric part of the gradient of u , i.e. $\tau = \tau(Du)$, where $Du = \frac{1}{2}[\nabla_x u + (\nabla_x u)^T]$. Function G_{ext} represents an external force.

The coupling of (1.1) with (1.2) is present in the drag force

$$F_d(t, x, v) := u(t, x) - v,$$

that influences the motion of particles and fluid. Explicitly, the coupled system reads as follows:

$$\begin{cases} \partial_t f + v \cdot \nabla_x f + \operatorname{div}_v[(F_{CS}(f) + F_d)f] &= 0, \\ \partial_t u + (u \cdot \nabla_x)u + \nabla_x \pi - \operatorname{div}_x(\tau) &= -3 \int_{\mathbb{R}^d} F_d f dv, \\ \operatorname{div}_x u &= 0. \end{cases} \quad (1.3)$$

The system is considered over the phase-space $\mathbb{T}_x^3 \times \mathbb{R}_v^3$ with a set of initial data. Our main result is presented by Theorem 2.1. It says, for any given initial velocity and distribution of particles assumed to be suitable regular there exists unique global in time regular solution, provided the growth of stress tensor $\tau(Du)$ is greater than $p - 1$ with $p \geq \frac{11}{5}$, the same as for the pure non-Newtonian fluid [27].

Let us briefly discuss the difference between coupling of the CS model with Newtonian and non-Newtonian fluids. In [4, 2], the authors obtained existence of weak solutions for

their coupled systems and on top of that in [2], the authors proved asymptotic flocking. In case of coupling with a non-Newtonian fluid, existence regularity and possibly uniqueness depend on the value of the exponent p and regularity of the external function G_{ext} . For the non-Newtonian system (1.2) existence of weak solutions is known for $p > \frac{2d}{d+2}$ and $G_{ext} \in L^p(0, T; (W^{1,p})^*(\mathbb{T}^d))$ and an important main tool in the proof is the Lipschitz truncation method [17, 15]. On the other hand, if $p \geq \frac{3d+2}{d+2}$ and $G_{ext} \in L^2(0, T; L^2(\mathbb{T}^d))$, we have not only existence of strong solutions but also their uniqueness [31].

The paper is organized as follows. First we introduce the system (1.3) and formulate the main results. In Section 3, the kernel of the paper, we prove the existence part of Theorem 2.1. Next, we show the uniqueness of obtained solutions. And finally in Appendix a number of auxiliary results are presented/proved.

2 Preliminaries

Let us introduce the notation. By $W^{k,p}(\Omega)$ we denote the Sobolev space of functions with up to the k -th weak derivative belonging to the Lebesgue space $L^p(\Omega)$. Moreover, $\mathcal{D}'(\Omega)$ denotes the space of distributions on Ω and $C^k(\Omega)$ is the space of the functions with up to the k -th derivative belonging to the space of continuous functions, which itself is denoted as $C(\Omega)$. We also use

$$A \stackrel{H(q)}{\leq} B$$

to emphasize that the estimate $A \leq B$ follows by Hölder's inequality with exponent q . We use a similar notation for Young's inequality replacing H with Y . An arbitrary generic constant is denoted by C ; its actual value may change depending on its appearances even in the same line.

Let us specify the structure of the main system (1.3). We start with explanation for the equations on motion of a non-Newtonian fluid. The sought elements are the velocity u and pressure p defined over the 3-dimensional periodic box $\mathbb{T}^3 = [0, 1]^3$ and the time interval $[0, T]$. For the stress tensor $\tau : \mathbb{R}_{sym}^{3^2} \rightarrow \mathbb{R}_{sym}^{3^2}$ there exist $p \in (1, \infty)$ and positive constants c_1 , c_2 and c_3 , such that for all $\xi, \eta \in \mathbb{R}_{sym}^{d^2}$

$$\tau_{ij}(\xi)\xi_{ij} \geq c_1(|\xi|^p + |\xi|^2), \quad |\tau_{ij}(\xi)| \leq c_2(1 + |\xi|)^{p-1}, \quad (2.1)$$

$$(\tau_{ij}(\xi) - \tau_{ij}(\eta))(\xi - \eta) \geq c_3(|\xi - \eta|^2 + |\xi - \eta|^p), \quad (2.2)$$

$$\frac{\partial \tau_{ij}(\eta)}{\partial \eta_{kl}} \xi_{ij} \xi_{kl} \geq c_4(1 + |\eta|)^{p-2} |\xi|^2, \quad \left| \frac{\partial \tau_{ij}(\eta)}{\partial \eta_{kl}} \right| \leq c_5(1 + |\eta|)^{p-2}. \quad (2.3)$$

As a classical example we point out $\tau(\xi) = C(1 + |\xi|)^{p-2}\xi$, keeping in mind that ξ is meant as the symmetric part of the velocity gradient, i.e. $\xi = Du = \frac{1}{2}(\nabla_x u + (\nabla_x u)^T)$.

Regarding the CS part of the system, we look for distribution function f defined over the phase-space $\mathbb{T}_x^3 \times \mathbb{R}_x^3$ for $t \in [0, T]$. The function is required to be non-negative. The equation on f is coupled to the fluid equations through the force term $F(f) = F_{CS}(f) + F_d$, where

$$F_d(t, x, v) := u(t, x) - v, \quad (2.4)$$

and

$$F_{CS}(f)(t, x, v) = \int_{\mathbb{T}^d \times \mathbb{R}^d} (w - v) \psi(|x - y|) f(t, y, w) dy dw \quad (2.5)$$

with $\psi(\cdot)$ — the communication weight — a non-negative, non-increasing and smooth function such that $\|\psi\|_{C^1} \leq c_6$. It follows $F_{CS}(f)(t, x, v) = a(t, x) - b(t, x)v$, with

$$a(t, x) := \int_{\mathbb{T}^3 \times \mathbb{R}^3} \psi(|x - y|) w f(t, y, w) dy dw, \quad (2.6)$$

$$b(t, x) := \int_{\mathbb{T}^3 \times \mathbb{R}^3} \psi(|x - y|) f(t, y, w) dy dw. \quad (2.7)$$

System (1.3) is supplemented by initial data u_0 and f_0 for the velocity field and distribution function, respectively.

For non-negative and integrable functions f we denote:

$$M_\alpha f(t) := \int_{\mathbb{T}^3 \times \mathbb{R}^3} |v|^\alpha f(t, x, v) dx dv, \quad m_\alpha f(t, x) := \int_{\mathbb{R}^3} |v|^\alpha f(t, x, v) dv,$$

with an obvious remark that $M_0 f = \|f\|_{L^1}$ and that for $1 \leq q \leq \infty$,

$$m_\alpha f(t, x) \leq C(R) \|f(t, x, \cdot)\|_q, \quad (2.8)$$

provided that $\text{supp} f(t, x, \cdot) \subset B(R)$, where $B(R)$ is a ball centered at 0 with radius R . Note that

$$\|a\|_\infty + \|\nabla_x a\|_\infty \leq c_6 M_1 f, \quad \|b\|_\infty + \|\nabla_x b\|_\infty \leq c_6 M_0 f, \quad (2.9)$$

hence

$$|F_{CS}(f)(t, x, v)| \leq \|a\|_\infty + |v| \|b\|_\infty \leq c_6 (M_1 f + |v| M_0 f), \quad (2.10)$$

and $\text{div}_v F_{CS}(f)(t, x, v) = -3b(t, x)$.

2.1 Weak formulation

Let us introduce the basic function spaces:

$$\begin{aligned} L_{div}^2(\mathbb{T}^3) &:= \{\omega \in L^2(\mathbb{T}^3) : \text{div}_x \omega = 0\}, \\ W_{div}^{1,p}(\mathbb{T}^3) &:= \{\omega \in \mathcal{D}'(\mathbb{T}^3) : \nabla_x \phi \in L^p(\mathbb{T}^3), \text{div}_x \omega = 0\}, \\ W_{div}^{1,2}(\mathbb{T}^3) &:= \{\omega \in W^{1,2}(\mathbb{T}^3) : \text{div}_x \omega = 0\}, \\ \mathcal{H} &:= L^\infty(0, T; \dot{W}_{div}^{1,p}(\mathbb{T}^3)) \cap C([0, T]; L_{div}^2(\mathbb{T}^3)) \cap L^2(0, T; W_{div}^{2,2}(\mathbb{T}^3)) \cap \\ &\quad \cap L^\infty(0, T; W_{div}^{1,2}(\mathbb{T}^3)) \cap L^p(0, T; \dot{W}_{div}^{1,3p}(\mathbb{T}^3)), \\ \mathcal{X} &:= L^\infty((0, T) \times \mathbb{T}^3 \times \mathbb{R}^3) \cap L^\infty(0, T; L^1(\mathbb{T}^3 \times \mathbb{R}^3)) \cap C([0, T]; L^2(\mathbb{T}^3 \times \mathbb{R}^3)) \cap \\ &\quad \cap L^\infty(0, T; W^{1,2}(\mathbb{T}^3 \times \mathbb{R}^3)). \end{aligned}$$

The spaces are endowed by the standard norms coming from definitions.

Next, we define weak solutions to (1.3).

Definition 2.1. Let $p \geq \frac{11}{5}$ and $T > 0$. The couple (f, u) is a weak solution of (1.3) on the time interval $[0, T]$ if and only if the following conditions are satisfied:

- (i) $f \geq 0$, $f \in \mathcal{X}$, $\partial_t f \in L^2(0, T; L^2(\mathbb{T}^3 \times \mathbb{R}^3))$, $\nabla_v f \in L^\infty(0, T; L^3(\mathbb{T}^3 \times \mathbb{R}^3))$ and $M_2 f \in L^\infty([0, T])$; the function $v \mapsto f(t, x, \cdot)$ is compactly supported for a.a. $t \in [0, T]$ and $x \in \mathbb{T}^3$.

(ii) $u \in \mathcal{H}$ and $\partial_t u \in L^2(0, T; L^2(\mathbb{T}^3))$.

(iii) For all $\phi \in C_b^1([0, T] \times \mathbb{T}^3 \times \mathbb{R}^3)$ such that $\phi|_{t=T} \equiv 0$, the following identity holds (the lower index b means that the function is bounded on $\mathbb{T}^3 \times \mathbb{R}^3$)

$$\int_0^T \int_{\mathbb{T}^3 \times \mathbb{R}^3} f[\partial_t \phi + v \cdot \nabla_x \phi + F(f) \cdot \nabla_v \phi] dx dv dt = - \int_{\mathbb{T}^3 \times \mathbb{R}^3} f_0 \phi(0, \cdot, \cdot) dx dv.$$

(iv) For all $\varphi \in W^{1,2}(\mathbb{T}^3) \cap \dot{W}_{div}^{1,p}(\mathbb{T}^3)$

$$\int_{\mathbb{T}^3} \left[\frac{\partial u}{\partial t} \cdot \varphi + (u \cdot \nabla_x) u \cdot \varphi + \tau(Du) : D(\varphi) \right] dx = -3 \int_{\mathbb{T}^3 \times \mathbb{R}^3} (u - v) \cdot \varphi f dx dv$$

is satisfied a.e. in $[0, T]$ and $\lim_{t \rightarrow 0^+} u(t, \cdot) = u_0$ in $L^2(\Omega)$.

Remark 2.1. In Definition 2.1, regularity of f and boundedness of the support of f in v enable us to rewrite point (iii) in an equivalent form:

(iii)' For all $\phi \in L_{loc}^\infty(\mathbb{T}^3 \times \mathbb{R}^3)$, we have

$$\int_{\mathbb{T}^3 \times \mathbb{R}^3} [\partial_t f + v \cdot \nabla_x f + \operatorname{div}_v [F(f)f]] \phi dx dv = 0$$

and $\lim_{t \rightarrow 0^+} \int_{\mathbb{T}^3 \times \mathbb{R}^3} f(t, \cdot, \cdot) \Phi dx dv = \int_{\mathbb{T}^3 \times \mathbb{R}^3} f_0 \Phi dx dv$ for all $\Phi \in C_b^1(\mathbb{T}^3 \times \mathbb{R}^3)$. Note that due to the fact that the derivatives of f are integrable and for a.a. $t \in [0, T]$, we have $f(t) \in L^\infty(\mathbb{T}^3 \times \mathbb{R}^3) \cap L^1(\mathbb{T}^3 \times \mathbb{R}^3)$, the function f is an admissible test function for its weak formulation. Similarly in case $f(t)$ has a compact support, $|v|^\alpha$ for $\alpha \geq 0$ is a good test function since it belongs to $L_{loc}^\infty(\mathbb{T}^3 \times \mathbb{R}^3)$.

2.2 Main result

We present the main results of the paper.

Theorem 2.1. Let $p \geq \frac{11}{5}$ and $T > 0$. Suppose that the initial data (f_0, u_0) satisfy

(i) $0 \leq f_0 \in (L^1 \cap L^\infty)(\mathbb{T}^3 \times \mathbb{R}^3) \cap W^{1,2}(\mathbb{T}^3 \times \mathbb{R}^3)$, $\operatorname{supp} f_0(x, \cdot) \subset B(R)$ for some $R > 0$ and a.a. $x \in \mathbb{T}^3$, where $B(R)$ is a ball centred at 0 with radius R ,

(ii) $u_0 \in W_{div}^{1,2}(\mathbb{T}^3)$,

(iii) $\nabla_v f_0 \in L^3(\mathbb{T}^3 \times \mathbb{R}^3)$.

Then there exists a unique solution of (1.3) in the sense of Definition 2.1 for regular communication weight ψ .

Remark 2.2. Assumption (i) in the above theorem immediately implies that $M_\alpha f_0 \leq C$ for some positive constant C and all $\alpha \geq 0$ (from the point of view of Definition 2.1 we need at least $M_2 f_0 \leq C$). Moreover, boundedness of the support of f_0 and assumption (iii) are needed only for the uniqueness. In order to obtain existence alone we could skip assumption (iii) and replace the boundedness of the support of f_0 by the assumption that $M_5 f_0 \leq C$.

3 Existence of solutions

Our first goal is to prove the existence part of Theorem 2.1. The proof follows closely the ideas of [4] and can be described by the following six parts:

1. First, we regularize system (1.3). For the particle part we regularize initial datum f_0 and F_d in (1.3)₁. Then we are allowed to use the standard method of characteristics, which needs sufficient regularity of the trajectories. For the fluid part, we include a cut-off function in v to the external force in (1.3)₂ and regularize it.
2. To solve the regularized system, we introduce an iterative scheme. We alternate between solving (1.3)₁ with F_d taken from previous iterations and (1.3)₂ with the external force defined by previous iterations.
3. For each iteration, we establish existence of solutions using standard techniques originating from [2, 31, 27].
4. Next, we converge with the iterations to the solution of the regularized system. Technique introduced in [4] is applied here. We need to control uniformly the support in v of the iterative solutions.
5. We converge with the solutions of the regularized system to a solution of (1.3) locally in time. We estimate the regularized solutions in \mathcal{X} and \mathcal{H} and apply the Aubin–Lions lemma to extract a convergent subsequence. The crucial role in these estimates is played by the estimate of $M_2 f_\epsilon + \frac{1}{3} \|u_\epsilon\|_2^2$ – the total kinetic energy of the system.
6. Lastly, we extend the local solution up to an arbitrary interval $[0, T]$, thus finishing the proof of existence.

3.1 Regularized system

Note first there is a need to control the support of f in v in the external force $G_{ext}(t, x) = \int_{\mathbb{R}^3} (u(t, x) - v) f(t, x, v) dv$. Introduce a cut-off function $\gamma_\epsilon : \mathbb{R}^3 \rightarrow \mathbb{R}$ such that the support in v of f is contained in a ball of the radius $\frac{1}{\epsilon}$. Then we define

$$G_\epsilon(t, x) = \int_{\mathbb{R}^3} (\theta_\epsilon * u(t, x) - v) \gamma_\epsilon(v) f(t, x, v) dv,$$

where $\gamma_\epsilon \in C^\infty(\mathbb{R}^3)$, $\text{supp } \gamma_\epsilon \subset B(1/\epsilon)$, $0 \leq \gamma_\epsilon \leq 1$, $\gamma_\epsilon = 1$ on $B(1/2\epsilon)$, $\gamma_\epsilon \rightarrow 1$ as $\epsilon \rightarrow 0^+$, and θ_ϵ is the standard mollifier i.e. $\theta_\epsilon(x) := \frac{1}{\epsilon^3} \theta\left(\frac{x}{\epsilon}\right)$, for some $0 \leq \theta \in C_0^\infty(\mathbb{T}^3)$ with $\int_{\mathbb{T}^3} \theta dx = 1$. We further regularize also the drag force in the CS equation. For $\epsilon > 0$ we denote the regularized force $F_\epsilon(f_\epsilon)$, where

$$F_\epsilon(f_\epsilon; u) := F_{CS}(f) + (\theta_\epsilon * u - v) \gamma_\epsilon.$$

We now write down the regularized system. For $\epsilon > 0$ we consider

$$\begin{cases} \partial_t f_\epsilon + v \cdot \nabla_x f_\epsilon + \text{div}_v [F_\epsilon(f_\epsilon; u_\epsilon) f_\epsilon] &= 0, \\ \partial_t u_\epsilon + (u_\epsilon \cdot \nabla_x) u_\epsilon + \nabla_x \pi_\epsilon - \text{div}_x (\tau(Du_\epsilon)) &= -3 \int_{\mathbb{R}^d} f_\epsilon (\theta_\epsilon * u_\epsilon - v) \gamma_\epsilon dv, \\ \text{div}_x u_\epsilon &= 0, \end{cases} \quad (3.1)$$

with a smooth, compactly supported (in the variable v in $B(R)$) initial data $f_{0,\epsilon}$, where $0 \leq f_{0,\epsilon} \rightarrow f_0$ strongly in $L^q(\mathbb{T}^3 \times \mathbb{R}^3)$, for all $1 < q < \infty$ and weakly $*$ in $L^\infty(\mathbb{T}^3 \times \mathbb{R}^3)$, $u_{0,\epsilon} = u_0$.

To solve the regularized problem (3.1) we apply the following **iterative scheme**. Denoting for notational simplicity $f^n := f_\epsilon^n$, $u^n := u_\epsilon^n$, we introduce the following scheme:

Initial step $n = 1$. We set $u^1(t, x) := u_0(x)$. Next, we solve the Cucker–Smale’s part of (1.3) with fixed u_1 :

$$\partial_t f^1 + v \cdot \nabla_x f^1 + \operatorname{div}_v [F_{CS}(f^1) + (\theta_\epsilon * u^1 - v) \gamma_\epsilon f^1] = 0,$$

with the initial datum $f^1(0, x, v) = f_{0,\epsilon}(x, v)$.

Inductive step. Suppose we have a well defined n -th solution (f^n, u^n) . Then we define u^{n+1} as the solution of the system

$$\begin{cases} \partial_t u^{n+1} + (u^{n+1} \cdot \nabla_x) u^{n+1} + \nabla_x \pi^{n+1} - \operatorname{div}_x \tau(u^{n+1}) &= G^{n+1} = -3 \int_{\mathbb{R}^d} f^n (\theta_\epsilon * u^n - v) \gamma_\epsilon dv, \\ \operatorname{div}_x u^{n+1} &= 0, \end{cases} \quad (3.2)$$

with the initial datum $u^{n+1}(0, x) = u_0(x)$ noting that in this system, the right-hand side depends on f^n and u^n , which are at this point given functions. Thus, in fact, we solve (1.2) with a given external force. Further, we define

$$F_d^{n+1} = (\theta_\epsilon * u^{n+1} - v) \gamma_\epsilon,$$

which is at this point a given function. Finally we solve the Vlasov–type equation:

$$\partial_t f^{n+1} + v \cdot \nabla_x f^{n+1} + \operatorname{div}_v [F_{CS}(f^{n+1}) + F_d^{n+1}] f^{n+1} = 0, \quad (3.3)$$

with initial datum $f^{n+1}(0, x, v) = f_{0,\epsilon}(x, v)$.

Existence of f^n and u^n is guaranteed by the following propositions belonging to the classical theory.

Proposition 3.1. *Let $p \geq \frac{11}{5}$ and $T > 0$. Then there exists a unique solution in the sense of Definition 2.1 to the problem*

$$\begin{aligned} \partial_t u + (u \cdot \nabla_x) u + \nabla_x \pi - \operatorname{div}_x (\tau(Du)) &= G, \\ \nabla_x \cdot u &= 0 \end{aligned}$$

provided $u_0 \in W_{0,div}^{1,2}(\mathbb{T}^3)$ and $G \in L^2(0, T; L^2(\mathbb{T}^3))$. Moreover,

$$\|u\|_{\mathcal{H}} \leq C, \quad \|\partial_t u\|_{L^2(0,T;L^2(\mathbb{T}^3))} \leq C,$$

where C is a positive constant depending on $\|u_0\|_{W^{1,2}(\mathbb{T}^3)}$, $\|G\|_{L^2(0,T;L^2(\mathbb{T}^3))}$, p and T .

Proof. The proof can be found in [27, Theorem 4.5]. It is based on the structure of $\tau(\cdot)$. We may consider different types of approximations for which it is not difficult to construct solutions. To obtain a priori estimates allowing to pass from the approximate problem to the original one, we first test the approximate problem by the velocity. The pressure and the convective term disappear due to the divergence-free condition and the time derivative and the stress tensor (property (2.1)) yield the estimates of the velocity in $L^\infty(0, T; L^2(\mathbb{T}^3))$ and in $L^p(0, T; W^{1,p}(\mathbb{T}^3))$. Next step consists in testing by $(1 + \|\nabla u(t)\|_{L^2(\mathbb{T}^3)}^{-\lambda}) \Delta u$ for suitable $\lambda \in [0, 1]$. The time derivative and the structure of the stress tensor (more precisely, property

(2.3)) provide now estimates in $L^\infty(0, T; W^{1,2}(\mathbb{T}^3))$, $L^p(0, T; W^{1,3p}(\mathbb{T}^3))$ and $L^2(0, T; W^{2,2}(\mathbb{T}^3))$. However, the convective term does not disappear now and we to control a term of the form $\sim |\nabla u|^3$ on the right hand-side, using estimates from the first step together with the form of the left-hand side. It is possible to estimate the cubic term for $p \geq \frac{11}{5}$ and get the following bound

$$\sup_{t < T} \|u\|_{W^{1,2}(\mathbb{T}^3)}^2 + \|\nabla_x^{(2)} u\|_{L^2(0,T;L^2(\mathbb{T}^3))}^2 + \|\nabla_x u\|_{L^p(0,T;L^{3p}(\mathbb{T}^3))}^p \leq C(\|G\|_{L^2(\mathbb{T}^3 \times (0,T))}^2 + \|u_0\|_{W^{1,2}(\mathbb{T}^3)}^2), \quad (3.4)$$

where the constant C depends on also on the estimates from the first step, i.e. on the norms of the velocity in $L^\infty(0, T; L^2(\mathbb{T}^3))$ and in $L^p(0, T; W^{1,p}(\mathbb{T}^3))$. Moreover, the velocity u can be used as a test function in the weak formulation which allows to exploit the monotone structure of the stress tensor in order to pass from the approximation to the original problem as well as to prove the uniqueness of the solution. Finally, using as a test function $\partial_t u$, we deduce the estimates of u in $L^p(0, T; W^{1,p}(\mathbb{T}^3))$ and $\partial_t u$ in $L^2((0, T) \times \mathbb{T}^3)$.

In a sense, we repeat the idea of the proof for the two dimension Navier–Stokes system, but with better integrability given by the features of $\tau(\cdot)$ for $p \geq 11/5$. \square

Proposition 3.2. *Let $T > 0$. There exists a solution in the sense of Definition 2.1 to the problem*

$$\partial_t f + v \cdot \nabla_x f + \operatorname{div}_v[(F_{CS}(f) + (\theta_\epsilon * u - v))\gamma_\epsilon(v)f] = 0, \quad (3.5)$$

as long as $0 \leq f_0 \in C^\infty(\mathbb{T}^3 \times \mathbb{R}^3)$ is compactly supported in v and $u \in L^\infty(0, T; L_{div}^2(\mathbb{T}^3))$. This solution f belongs to the space $C^2([0, T] \times \mathbb{T}^3 \times \mathbb{R}^3)$. Moreover

$$\|f\|_{L^\infty(0,T;(L^\infty \cap L^1)(\mathbb{T}^3 \times \mathbb{R}^3))} \leq C, \quad \|f\|_{C^2} \leq C(\epsilon), \quad (3.6)$$

where C is a positive constant depending on $\|f_0\|_{L^1(\mathbb{T}^3 \times \mathbb{R}^3)}$ and $\|f_0\|_{L^\infty(\mathbb{T}^3 \times \mathbb{R}^3)}$, while $C(\epsilon)$ depends also on ϵ and $\|u\|_{L^\infty(0,T;L^2(\mathbb{T}^3))}$ (both constants depend also on T). Furthermore, $f \geq 0$ in $[0, T] \times \mathbb{T}^3 \times \mathbb{R}^3$, provided $f_0 \geq 0$ in $\mathbb{T}^3 \times \mathbb{R}^3$.

Proof. This proposition along with its proof can be found in [2, Appendix A]. However, there is one seemingly substantial difference, namely in [2], it is only shown that $f \in C^1$. It can be obtained by an easy modification of the proof from [2], since both $F_{CS}(f)$ and F_d in (3.5) are smooth. Local existence in Proposition 3.2 is showed by a standard method of characteristics combined with a fixed point argument. Then to conclude the global existence, a priori C^2 estimate for f is derived. It can be done because the nonlinearity in (3.5) that comes from the multiplication by $F_{CS}(f) + F_d$ is smooth (here regularity of the communication weight ψ and the mollifier θ_ϵ play the crucial role). \square

Let us now return to our sequence of approximate solutions. u^n and f^n exist in the sense of Definition 2.1 (see Proposition 3.2). Then $u^n \in L^\infty(0, T; L_{div}^2(\mathbb{T}^3))$ and $f^n \in C^2([0, T] \times \mathbb{T}^3 \times \mathbb{R}^3)$ what follows

$$\begin{aligned} \int_0^T \|G^{n+1}\|_{L^2(\mathbb{T}^3)}^2 &= 9 \int_0^T \int_{\mathbb{T}^3} \left| \int_{\mathbb{R}^3} (\theta_\epsilon * u^n - v) \gamma_\epsilon f^n dv \right|^2 dx dt \\ &= 9 \int_0^T \int_{\mathbb{T}^3} \left| \int_{B(\frac{1}{\epsilon})} (\theta_\epsilon * u^n - v) \gamma_\epsilon f^n dv \right|^2 dx dt \\ &\leq C(T, \epsilon) \int_0^T \int_{\mathbb{T}^3} \int_{B(\frac{1}{\epsilon})} |\theta_\epsilon * u^n - v|^2 |f^n|^2 dv dx dt \\ &\leq C(T, \epsilon) \|f^n\|_{L^\infty((0,T) \times \mathbb{T}^3 \times \mathbb{R}^3)}^2 (\|u^n\|_{L^2((0,T) \times \mathbb{T}^3)}^2 + 1). \end{aligned} \quad (3.7)$$

Therefore G^{n+1} belongs to $L^2(0, T; L^2(\mathbb{T}^3))$ with its norm depending on $T, \epsilon, \|f^n\|_{L^\infty((0, T) \times \mathbb{T}^3 \times \mathbb{R}^3)}$ and $\|u\|_{L^\infty(0, T; L^2(\mathbb{T}^3))}$. Therefore, by Proposition 3.1, there exists a unique u^{n+1} — a solution to (3.2) in the sense of Definition 2.1. Existence of a unique f^{n+1} — a solution to (3.3) belonging additionally to the space $C^2([0, T] \times \mathbb{T}^3 \times \mathbb{R}^3)$ — follows then by Proposition 3.2. This argument may be iterated indefinitely and thus, the sequence (f^n, u^n) is well defined.

3.2 Step 4: Convergence of the iterations

Our next step is to prove that with $n \rightarrow \infty$, functions (f^n, u^n) converge¹ to a solution of (3.1). We begin with estimates for u^n and f^n in \mathcal{H} and \mathcal{X} , respectively.

Proposition 3.3. *Sequence $\{u^n, f^n\}$, the approximate solutions, satisfies the following bounds:*

$$\begin{aligned} (i) \quad & \|u^n\|_{\mathcal{H}} \leq C(\epsilon), \\ (ii) \quad & \|\partial_t u^n\|_{L^2(0, T; L^2(\mathbb{T}^3))} \leq C(\epsilon), \\ (iii) \quad & \|\nabla_v f^n\|_{L^\infty(0, T; L^3(\mathbb{T}^3 \times \mathbb{R}^3))} \leq C, \\ (iv) \quad & \|f^n\|_{\mathcal{X}} \leq C(\epsilon), \\ (v) \quad & \|\partial_t f^n\|_{L^2(0, T; L^2(\mathbb{T}^3 \times \mathbb{R}^3))} \leq C(\epsilon), \end{aligned}$$

where $C(\epsilon)$ is independent of n but depends on ϵ , while C in (iii) is independent of n and ϵ . Moreover, there exists a non-decreasing function $\mathcal{R}_\epsilon : [0, T] \rightarrow [0, \infty)$ such that

$$\text{supp } f^n(t, x, \cdot) \subset B(\mathcal{R}_\epsilon(t)), \quad \text{for all } t \text{ and a.a } x. \quad (3.8)$$

The function \mathcal{R}_ϵ is independent of n but may depend on ϵ and R .

In what follows, we use the following notation. The norm $\|\cdot\|_q$ denotes the L^q -norm, either over \mathbb{T}^3 or over $\mathbb{R}^3 \times \mathbb{T}^3$, in dependence on the function which norm we have in mind. In case it will be necessary to distinguish, we will use the full notation of the norm. The same holds in case we consider also the time variable.

Proof. By Proposition 3.1 and the definition of u^n it is clear that to obtain estimation of u^n in \mathcal{H} it suffices to estimate $\|G^{n+1}\|_{L^2(0, T; L^2(\mathbb{T}^3))}$ uniformly with respect to n . By testing the weak formulation by u^n (which by Proposition 3.1 is a suitable test function), applying Korn's inequality and (2.1) we obtain

$$\frac{1}{2} \frac{d}{dt} \|u^n\|_2^2 + c_1 \kappa \|\nabla_x u^n\|_p^p \leq \|u^n\|_2^2 + \|G^{n+1}\|_2^2,$$

which by inequality (3.7) and (3.6) implies that

$$\frac{1}{2} \frac{d}{dt} \|u^n\|_2^2 + c_1 \kappa \|\nabla_x u^n\|_p^p \leq \|u^n\|_2^2 + C(T, \epsilon) \|u^{n-1}\|_2^2.$$

Therefore by Lemma A.2 we have

$$\|u^n\|_{L^\infty(0, T; L^2(\mathbb{T}^3))} + \|\nabla_x u^n\|_{L^p((0, T) \times \mathbb{T}^3)} \leq C(T, \epsilon),$$

which together with (3.7) proves (by induction) that the right-hand side G^{n+1} is uniformly bounded in $L^2(0, T; L^2(\mathbb{T}^3))$. Using Proposition 3.1 we finish the proof of (i).

¹Keep in mind that we skip the epsilon in $(f^n, u^n) = (f_\epsilon^n, u_\epsilon^n)$.

The proof of (ii) follows similarly to the proof of (i) by testing the weak formulation for u^n with $\partial_t u^n$ and using the previously proved estimates.

We continue with estimates of f^n .

Proof of the propagation of the support. The estimate of the support of f^n is proved in Lemmas 3.1 and 3.2 below. Lemma 3.1 shows that $\mathcal{R}_\epsilon(t)$ depends on $\|u^n\|_{L^2(0,T;W^{2,2}(\mathbb{T}^3))}$ and $\|M_1 f^n\|_{L^\infty(0,T)}$. On the other hand, in Lemma 3.2 we prove that $\|M_1 f^n\|_{L^\infty(0,T)}$ is uniformly bounded in terms of $\|u^n\|_{L^2(0,T;W^{2,2}(\mathbb{T}^3))}$. Therefore, by (i) from Proposition 3.3, the function \mathcal{R}_ϵ is independent of n but depends on ϵ .

This observation concludes the proof of (3.8).

Lemma 3.1 (Propagation of the support of velocity). *Let f be a solution to (3.5) subjected to the initial data with the support in v contained in the ball $B(R)$. Then there exists a non-decreasing function $\mathcal{R} : [0, T] \rightarrow [0, \infty)$ such that for all $t \in [0, T]$ and almost all $x \in \mathbb{T}^3$, the support of $f(t, x, \cdot) : \mathbb{R}^3 \rightarrow \mathbb{R}$ is contained in a ball of radius $\mathcal{R}(t)$. Moreover, for each $t \in [0, T]$ the value $\mathcal{R}(t)$ depends only on t , $\|u\|_{L^2(0,t;W^{2,2}(\mathbb{T}^3))}$, $\|M_1 f\|_{L^\infty(0,T)}$ and R .*

Proof. Let f be a solution to (3.5). Consider the solution of the system of ODE's:

$$\begin{cases} \frac{dx}{dt}(t) = v(t), & x(0) = x_0, \\ \frac{dv}{dt}(t) = F_{CS}(f)(t, x(t), v(t)) + [(\theta_\epsilon * u)(t, x(t)) - v(t)]\gamma_\epsilon(v(t)), & v(0) = v_0. \end{cases} \quad (3.9)$$

Then the function $\tilde{f}(t, x_0, v_0) := f(t, x(t), v(t))$ satisfies the equation

$$\partial_t \tilde{f} = (-\operatorname{div}_v F_{CS}(f) + \operatorname{div}_v [(\theta_\epsilon * u(t, x(t)) - v)\gamma_\epsilon(v)])\tilde{f}.$$

Note that we are required to look at the three terms coming from the divergence: $\gamma_\epsilon(v)$, $\theta_\epsilon * u \nabla_v \gamma_\epsilon(v)$ and $v \cdot \nabla_v \gamma_\epsilon(v)$. By definition of $\gamma_\epsilon(\cdot)$ from the beginning of Section 3.1 we see that

$$|\gamma_\epsilon(v)| + |v \cdot \nabla_v \gamma_\epsilon(v)| \leq C$$

with C independent of ϵ . On other hand, using the explicit form of θ_ϵ , it is possible to compute that $|\theta_\epsilon * u| \leq \|u(t, \cdot)\|_{L^\infty} \leq C\epsilon^{-3/8}\|u(t, \cdot)\|_{W_{11/5}^1}$, so we conclude

$$|\theta_\epsilon * u \nabla_v \gamma_\epsilon(v)| \leq C\epsilon^{5/8}.$$

Hence, recalling b is defined in (2.7), we obtain

$$\tilde{f}(t, x(t), v(t)) = e^{3 \int_0^t (b+B)ds} f_0(x(t), v(t)), \quad \text{where } \|B(t)\|_{L^\infty} \leq C(\|u(t, \cdot)\|_{W_p^1} + C) \quad (3.10)$$

for sufficiently small ϵ .

Therefore, $\tilde{f}(t, x_0, v_0) = 0$ whenever $f_0(x_0, v_0) = 0$ which implies that $f(t, x, v) = 0$ whenever the characteristic that contains point (x, v) starts at (x_0, v_0) such that $f_0(x_0, v_0) = 0$. We solve (3.9)₂, to get

$$\begin{aligned} v(t) &= e^{-\int_0^t (b(s, x(s)) + B(s, x(s), v(s)))ds} \times \\ &\times \left(v_0 + \int_0^t e^{\int_s^t (b(r, x(r)) + B(s, x(s), v(s)))dr} [a(s, x(s)) + (\theta_\epsilon * u)(s, x(s))] \gamma_\epsilon(v(s)) ds \right), \end{aligned}$$

which, since by (2.9) $1 \leq b + 1 \leq c_6 M_0 f + 1$, by (3.6) $M_0 f \leq C$, and B is bounded in terms of the norm $L^2(0, T; W^{2,2}(\mathbb{T}^3))$, implies that

$$\begin{aligned} |v(t)| &\leq C e^{Ct} \left(|v_0| + \int_0^t |a(x(s), s)| ds + \int_0^t \|u(s)\|_\infty ds \right) \\ &\leq C e^{Ct} \left(|v_0| + t \|M_1 f\|_{L^\infty(0, T)} + \|u\|_{L^2(0, t; W^{2,2}(\mathbb{T}^3))} \right) \\ &\leq C e^{Ct} \left(R + t \|M_1 f\|_{L^\infty(0, T)} + \|u\|_{L^2(0, T; W^{2,2}(\mathbb{T}^3))} \right) =: \mathcal{R}(t), \end{aligned}$$

where we also used the embedding $L^2(0, t; W^{2,2}(\mathbb{T}^3)) \hookrightarrow L^1(0, t; L^\infty(\mathbb{T}^3))$. We will underline at this stage that the estimate depends of ϵ , but at the end of the proof the control of norms for u , this estimate will imply that the support of f is bounded independently of ϵ . \square

Lemma 3.2. *Let f be a solution to (3.5) subjected to the initial data with the support in v contained in the ball $B(R)$. Then*

$$M_1 f \leq C(\epsilon),$$

for some positive ϵ -dependent constant $C(\epsilon)$.

Proof. First we integrate (3.5) to see that $M_0 f = \text{const.}$ Next we multiply (3.5) by $|v|$ and integrate to get

$$\begin{aligned} 0 &= \frac{d}{dt} M_1 f + \underbrace{\int_{\mathbb{T}^3 \times \mathbb{R}^3} |v| v \cdot \nabla_x f dx dv}_{=0} + \int_{\mathbb{T}^3 \times \mathbb{R}^3} |v| \text{div}_v \left[(F_{CS}(f) + (\theta_\epsilon * u - v) \gamma_\epsilon) f \right] dx dv \\ &= \frac{d}{dt} M_1 f - \int_{\mathbb{T}^3 \times \mathbb{R}^3} \frac{v}{|v|} \cdot (F_{CS}(f) + (\theta_\epsilon * u - v) \gamma_\epsilon) f dx dv. \end{aligned}$$

Thus

$$\begin{aligned} \frac{d}{dt} M_1 f &= \int_{\mathbb{T}^3 \times \mathbb{R}^3} \frac{v}{|v|} \cdot F_{CS}(f) f dx dv + \int_{\mathbb{T}^3 \times \mathbb{R}^3} \frac{v}{|v|} \cdot \theta_\epsilon * u \gamma_\epsilon f dx dv - \int_{\mathbb{T}^3 \times \mathbb{R}^3} |v| f \gamma_\epsilon dx dv \\ &\leq \int_{\mathbb{T}^3 \times \mathbb{R}^3} |F_{CS}(f)| f dx dv + \int_{\mathbb{T}^3 \times \mathbb{R}^3} |\theta_\epsilon * u| f dx dv \\ &\stackrel{(2.10)}{\leq} C M_1 f M_0 f + C \|u\|_{W^{2,2}(\mathbb{T}^3)} M_0 f. \end{aligned}$$

Since $M_0 f = \text{const.}$, it implies that $M_1 f$ is bounded on $[0, T]$ if $M_1 f_0$ is finite. \square

Remark 3.1. In the proof of (iii) and (iv) we estimate a and b from (2.9) in $W^{1,\infty}$. It follows by the fact $M_1 f^n$ and $M_0 f^n$ are by (3.6) uniformly bounded with respect to n (in fact $M_0 f^n$ is uniformly bounded also with respect to ϵ).

Proof of (iii). We apply ∇_v to both sides of (3.5), multiply by $|\nabla_v f^n| \nabla_v f^n$ and integrate,

obtaining

$$\begin{aligned}
& -\frac{1}{3} \frac{d}{dt} \|\nabla_v f^n\|_3^3 \\
& = \int_{\mathbb{T}^3 \times \mathbb{R}^3} \nabla_v [\operatorname{div}_v F_{CS}(f^n) f^n + F_{CS}(f^n) \nabla_v f^n - 3f^n + (\theta_\epsilon * u^n - v) \gamma_\epsilon \nabla_v f^n] |\nabla_v f^n| \nabla_v f^n dx dv \\
& \stackrel{(2.9)}{=} \int_{\mathbb{T}^3 \times \mathbb{R}^3} \nabla_v [-3bf^n + (a - vb) \nabla_v f^n - 3f^n + (\theta_\epsilon * u^n - v) \cdot \gamma_\epsilon \nabla_v f^n] |\nabla_v f^n| \nabla_v f^n dx dv \\
& = \int_{\mathbb{T}^3 \times \mathbb{R}^3} (-4b - 4) |\nabla_v f^n|^3 dx dv + \int_{B(1/\epsilon) \times \mathbb{R}^3} [(a - vb) + (\theta_\epsilon * u^n - v) \gamma_\epsilon] \cdot \nabla_v^{(2)} f^n \cdot |\nabla_v f^n| \nabla_v f^n dx dv \\
& + \int_{B(1/\epsilon) \times \mathbb{R}^3} (\theta_\epsilon * u^n - v) \nabla_v \gamma_\epsilon : \nabla_v f^n \nabla_v f^n |\nabla_v f^n| dx dv =: \mathcal{L}.
\end{aligned}$$

The second integral appearing in the above definition of \mathcal{L} can be rewritten as

$$\int_{\mathbb{T}^3 \times \mathbb{R}^3} [(a - vb) + (\theta_\epsilon * u^n - v)] \gamma_\epsilon \frac{1}{3} \nabla_v |\nabla_v f^n|^3 dx dv,$$

which after integrating by parts² is bounded by $2 \int_{\mathbb{T}^3 \times \mathbb{R}^3} (b + 1) |\nabla_v f^n|^3 dx dv$. Thus

$$\mathcal{L} \leq -3 \int_{\mathbb{T}^3 \times \mathbb{R}^3} (b + 1) |\nabla_v f^n|^3 dx dv \stackrel{(2.9)}{\leq} (c_6 + 1) \|\nabla_v f^n\|_3^3.$$

Altogether, we get

$$\frac{1}{3} \frac{d}{dt} \|\nabla_v f^n\|_3^3 \leq (c_6 + 1) \|\nabla_v f^n\|_3^3,$$

hence by Gronwall's lemma for $t \in [0, T]$ we have $\|\nabla_v f^n(t)\|_3^3 \leq C \|\nabla_v f_0\|_3^3$, which together with assumption (iii) from Theorem 2.1 finishes the proof of estimate (iii). Recall all constants depend on T , but they are independent of ϵ and n .

Proof of (iv). The proof of boundedness of $\nabla_v f^n$ in $L^\infty(0, T; L^2(\mathbb{T}^3 \times \mathbb{R}^3))$ follows exactly like the above proof of (iii) (we just test with $\nabla_v f^n$ instead of $|\nabla_v f^n| \nabla_v f^n$ and use assumption (i) from Theorem 2.1). Therefore in order to estimate f^n in X , we only need to estimate $\nabla_x f^n$ in $L^\infty(0, T; L^2(\mathbb{T}^3 \times \mathbb{R}^3))$. Note that in the following estimate we do not use the mollifying effect of θ_ϵ ; we do not need this effect at this point and on top of that ultimately we need ϵ independent estimates. To estimate $\nabla_x f^n$, we apply ∇_x to both sides of (3.5), multiply by $\nabla_x f^n$ and integrate to obtain (similarly to the previous step)

$$\begin{aligned}
& -\frac{1}{2} \frac{d}{dt} \|\nabla_x f^n\|_2^2 = \int_{\mathbb{T}^3 \times \mathbb{R}^3} \nabla_x [-3bf^n + (a - vb) \nabla_v f^n - 3f^n + (\theta_\epsilon * u^n - v) \gamma_\epsilon \nabla_v f^n] \cdot \nabla_x f^n dx dv \\
& = \int_{\mathbb{T}^3 \times \mathbb{R}^3} -3 \nabla_x b \cdot \nabla_x f^n f^n dx dv + \int_{\mathbb{T}^3 \times \mathbb{R}^3} [\nabla_x a - v \nabla_x b + \gamma_\epsilon \theta_\epsilon * \nabla_x u^n] : \nabla_v f^n \nabla_x f^n dx dv \\
& + \int_{\mathbb{T}^3 \times \mathbb{R}^3} [a - vb + (\theta_\epsilon * u^n - v) \gamma_\epsilon] \cdot (\nabla_x \nabla_v f^n \nabla_x f^n) dx dv + \int_{\mathbb{T}^3 \times \mathbb{R}^3} (-3 - 3b) |\nabla_x f^n|^2 dx dv \\
& =: I + II + III + IV.
\end{aligned}$$

Then by (2.7), (3.6) and Young's inequality we have $|I| \leq C(1 + \|\nabla_x f^n\|_2^2)$. To estimate II , we use Young's inequality together with (2.9) and the estimate of the support in v of f to get

$$|II| \leq C(T, \epsilon) \left(\|\nabla_v f^n\|_2^2 + \|\nabla_x f^n\|_2^2 + \|\nabla_v f^n\|_3^3 + \|\theta_\epsilon * \nabla_x u^n \nabla_x f^n\|_{\frac{3}{2}}^{\frac{3}{2}} \right).$$

²Keeping in mind that $\operatorname{supp} f(x, \cdot, t) \subset B(\mathcal{R}_\epsilon(T))$.

By Hölder's inequality with exponent $\frac{4}{3}$ and Young's inequality for convolution we have

$$|II| \leq C(T, \epsilon) \left(\|\nabla_v f^n\|_2^2 + \|\nabla_x f^n\|_2^2 + \|\nabla_v f^n\|_3^3 + \|\nabla_x u^n\|_6^2 \|\nabla_x f^n\|_2^2 \right)$$

which together with (iii) and the boundedness of the support of f^n implies that

$$|II| \leq C(T, \epsilon) \left(1 + \|\nabla_x f^n\|_2^2 + \|\nabla_x u^n\|_6^2 \|\nabla_x f^n\|_2^2 \right).$$

Finally for III we integrate by parts to obtain

$$III = \frac{1}{2} \int_{\mathbb{T}^3 \times \mathbb{R}^3} (b+1) |\nabla_x f^n|^2 dx dv \leq c_6 \|\nabla_x f^n\|_2^2$$

and altogether

$$\frac{1}{2} \frac{d}{dt} \|\nabla_x f^n\|_2^2 \leq C(T, \epsilon) \left(1 + \|\nabla_x f^n\|_2^2 + \|\nabla_x u^n\|_6^2 \|\nabla_x f^n\|_2^2 \right)$$

which by Gronwall's lemma and estimate (i) proves (iv) (we also use the assumption on the initial data (i) from Theorem 2.1).

Proof of (v). We multiply both sides of (3.5) by $\partial_t f^n$ and integrate. Using estimates (i)–(iv) and Young's inequality we finish the proof. \square

With the estimates provided by Proposition 3.3 we are ready to prove convergence of (f^n, u^n) to a solution of the regularised system (3.1). First, let us denote for $n = 1, 2, \dots$

$$\begin{cases} \chi^n(t) &:= (x^n(t), v^n(t)), \\ \omega^{n+1} &:= u^{n+1} - u^n, \end{cases} \quad (3.11)$$

where $(x^n(t), v^n(t))$ is a solution of the following characteristics ODE:

$$\begin{cases} \frac{dx^n}{dt}(t) = v^n(t), & x^n(0) = x, \\ \frac{dv^n}{dt}(t) = F_{CS}(f^n)(x^n(t), v^n(t), t) + \gamma_\epsilon(v(t))(\theta_\epsilon * u^n)(x^n(t), t) - v^n(t), & v^n(0) = v. \end{cases}$$

The following proposition proves the existence of the solution to the regularized system (3.1) and finishes the part 4 of the proof of Theorem 2.1.

Proposition 3.4. *For each $n = 1, 2, \dots$, there exists a unique solution (f^n, u^n) of (3.2) and (3.3) in the sense of Definition 2.1. The sequence (f^n, u^n) converges strongly in $L^\infty([0, T] \times \mathbb{T}^3 \times \mathbb{R}^3) \times L^\infty(0, T; W^{1,2}(\mathbb{T}^3))$ towards the weak formulation of (3.1). Such solution of (3.1) satisfies additionally inequality (3.6) and estimates (i)–(v) and (3.8) from Proposition 3.3.*

Proof. Existence of (f^n, u^n) belonging to the appropriate spaces was already explained with the help of Proposition 3.1 and Proposition 3.2. It remains to show that (f^n, u^n) converges strongly in $L^\infty([0, T] \times \mathbb{T}^d \times \mathbb{R}^d) \times L^\infty(0, T; W^{1,2}(\mathbb{T}^d))$ to the weak formulation of (3.1). To achieve this goal, let us first prove the following lemma.

Lemma 3.3. *For a given positive T and ϵ , let (f^n, u^n) be the n -th solution of (3.2) and (3.3). Then for $t \in [0, T]$, we have*

$$(1) \quad \|f^n(t) - f^{n-1}(t)\|_\infty + \|\chi^n(t) - \chi^{n-1}(t)\|_\infty \leq C(\epsilon) \int_0^T \|\omega^n(s)\|_2 ds,$$

$$(2) \quad \|\omega^{n+1}(t)\|_2^2 + \int_0^t \|\nabla_x \omega^{n+1}(s)\|_2^2 ds \leq C(\epsilon) \left(\int_0^t \|\omega^n(s)\|_2^2 ds + \int_0^t \|\omega^{n+1}(s)\|_2^2 ds \right),$$

where $C(\epsilon)$ is a positive constant independent of n , ω^n and χ^n are defined in (3.11).

Proof. Since the CS part of the system is exactly the same as in [2] and the mollifier θ_ϵ makes up for any differences that could appear due to different fluid part of the system, the proof of (1) is the same as in [2], which leaves us with only point (2) to prove. By Proposition 3.1, ω^{n+1} is a good test function for the weak formulation for ω^{n+1} , i.e.

$$\begin{aligned} & \int_{\mathbb{T}^3} \partial_t \omega^{n+1} \cdot \omega^{n+1} dx + \int_{\mathbb{T}^3} (\omega^{n+1} \cdot \nabla_x) u^{n+1} \cdot \omega^{n+1} dx + \int_{\mathbb{T}^3} (u^n \cdot \nabla_x) \omega^{n+1} \cdot \omega^{n+1} dx \\ & + \int_{\mathbb{T}^3} [\tau(Du^{n+1}) - \tau(Du^n)] : D\omega^{n+1} dx \\ & = -3 \int_{\mathbb{T}^3 \times \mathbb{R}^3} \gamma_\epsilon f^n(u^n - v) \cdot \omega^{n+1} dx dv - 3 \int_{\mathbb{T}^3 \times \mathbb{R}^3} \gamma_\epsilon f^{n-1}(u^{n-1} - v) \cdot \omega^{n+1} dx dv. \end{aligned}$$

Let us denote the convective term by

$$T_1 := \int_{\mathbb{T}^3} (\omega^{n+1} \cdot \nabla_x) u^{n+1} \cdot \omega^{n+1} dx + \int_{\mathbb{T}^3} (u^n \cdot \nabla_x) \omega^{n+1} \cdot \omega^{n+1} dx,$$

the stress tensor term by $T_2 := \int_{\mathbb{T}^3} [\tau(Du^{n+1}) - \tau(Du^n)] : D\omega^{n+1} dx$ and the drag force term by

$$T_3 := -3 \int_{\mathbb{T}^3 \times \mathbb{R}^3} \gamma_\epsilon f^n(u^n - v) \cdot \omega^{n+1} dx dv - 3 \int_{\mathbb{T}^3 \times \mathbb{R}^3} \gamma_\epsilon f^{n-1}(u^{n-1} - v) \cdot \omega^{n+1} dx dv.$$

First let us note that by (2.2) and Korn's inequality, we have

$$T_2 = \int_{\mathbb{T}^3} [\tau(Du^{n+1}) - \tau(Du^n)] : D\omega^{n+1} dx \geq c_3 \kappa \|\nabla_x \omega^{n+1}\|_2^2. \quad (3.12)$$

Next we focus on T_1 . By space periodicity and equation (3.2)₂ the second summand in T_1 equals 0. Hence

$$|T_1| = \left| \int_{\mathbb{T}^d} (\omega^{n+1} \cdot \nabla_x) u^{n+1} \cdot \omega^{n+1} dx \right| \stackrel{H(2)}{\leq} \|\omega^{n+1}\|_4^2 \|\nabla_x u^{n+1}\|_2$$

which by interpolation inequality and embeddings implies that

$$|T_1| \leq \|\omega^{n+1}\|_2^{\frac{1}{2}} \|\omega^{n+1}\|_6^{\frac{3}{2}} \|\nabla_x u^{n+1}\|_2 \stackrel{Y(4)}{\leq} C(\epsilon) \|\omega^{n+1}\|_2^2 + \frac{c_3 \kappa}{2} \|\nabla_x \omega^{n+1}\|_2^2, \quad (3.13)$$

where we also used (ii) from Proposition 3.3, $\|\nabla_x u^{n+1}(t)\|_2 \leq C(\epsilon)$ for all n . Lastly, for T_3

$$\begin{aligned} T_3 &= -3 \int_{\mathbb{T}^3 \times \mathbb{R}^3} \gamma_\epsilon (f^n - f^{n-1})(u^n - v) \cdot \omega^{n+1} dx dv - 3 \int_{\mathbb{T}^3 \times \mathbb{R}^3} \gamma_\epsilon f^{n-1}(u^n - u^{n-1}) \cdot \omega^{n+1} dx dv \\ &=: T_{31} + T_{32}, \end{aligned}$$

where

$$\begin{aligned} \frac{1}{3} |T_{31}| &\stackrel{Y(2)}{\leq} \frac{1}{2} \int_{\mathbb{T}^3 \times B(\frac{1}{\epsilon})} |f^n - f^{n-1}|^2 |u^n - v|^2 dx dv + \frac{1}{2} \int_{\mathbb{T}^3 \times B(\frac{1}{\epsilon})} |\omega^{n+1}|^2 dx dv \\ &\leq C(\epsilon) \|f^n - f^{n-1}\|_\infty^2 (\|u^n\|_2^2 + 1) + C(\epsilon) \|\omega^{n+1}\|_2^2, \\ \frac{1}{3} |T_{32}| &\stackrel{Y(2)}{\leq} C(\epsilon) \|f^{n-1}\|_\infty (\|\omega^n\|_2^2 + \|\omega^{n+1}\|_2^2), \end{aligned}$$

which together with (3.6) and (i) from Proposition 3.3 implies that

$$|T_3| \leq C(\epsilon) \left(\|f^n - f^{n-1}\|_\infty^2 + \|\omega^{n+1}\|_2^2 + \|\omega^n\|_2^2 \right). \quad (3.14)$$

Combining together estimates (3.12), (3.13), (3.14) and estimate from point (1) we obtain

$$\frac{1}{2} \frac{d}{dt} \|\omega^{n+1}\|_2^2 + \frac{c_3 \kappa}{2} \|\nabla_x \omega^{n+1}\|_2^2 \leq C(\epsilon) \left(\int_0^t \|\omega^n\|_2^2(s) ds + \|\omega^{n+1}\|_2^2 + \|\omega^n\|_2^2 \right)$$

and by integration of the previous inequality, having in mind that

$$\int_0^t \int_0^s \|\omega^n(r)\|_2^2 dr ds \leq T \int_0^t \|\omega^n(r)\|_2^2 dr,$$

we obtain point (2). \square

We are now sufficiently equipped to finish the proof of Proposition 3.4. By Lemma A.2 and Lemma 3.3.2 there exists K , such that

$$\|u^n(t) - u^{n-1}(t)\|_2^2 = \|\omega^n(t)\|_2^2 \leq \frac{K^n t^n}{n!}.$$

Thus

$$u^n \rightarrow u \quad \text{in } L^\infty(0, T; L^2(\mathbb{T}^3)) \quad (3.15)$$

for some $u \in L^\infty(0, T; L^2(\mathbb{T}^3))$ and due to Lemma 3.3 (2)

$$\nabla_x u^n \rightarrow \nabla_x u \quad \text{in } L^2(0, T; L^2(\mathbb{T}^3)). \quad (3.16)$$

Moreover, by Lemma 3.3 (1) it follows that

$$f^n \rightarrow f \quad \text{in } L^\infty([0, T] \times \mathbb{T}^3 \times \mathbb{R}^3) \quad (3.17)$$

for some $f \in L^\infty([0, T] \times \mathbb{T}^3 \times \mathbb{R}^3)$.

To finish the proof we need to show that (f, u) satisfies (3.1) in the sense of Definition 2.1. By (3.15)

$$\int_0^T \int_{\mathbb{T}^3} -u^n \cdot \partial_t \phi dx dv dt \rightarrow \int_0^T \int_{\mathbb{T}^3} -u \cdot \partial_t \phi dx dv dt, \quad (3.18)$$

for all divergence free smooth ϕ with compact support in t . Thus $\partial_t u^n \rightarrow \partial_t u$ in the distributional sense, where $\partial_t u$ is the distributional derivative of u . However, since $\partial_t u^n$ is bounded in $L^2(0, T; L^2(\mathbb{T}^3))$, it actually implies that $\partial_t u^n \rightarrow \partial_t u$ weakly in $L^2(0, T; L^2(\mathbb{T}^3))$.

By (3.15) and (3.16) $u^n \rightarrow u$ and $\nabla_x u^n \rightarrow \nabla_x u$ a.e. which implies that also the convective term $(u^n \cdot \nabla_x) u^n \rightarrow (u \cdot \nabla_x) u$ a.e. Moreover, for a sufficiently small $\eta > 0$ we have

$$\|(u^n \cdot \nabla_x) u^n\|_{L^{1+\eta}(0, T; L^{1+\eta}(\mathbb{T}^3))} \leq \|u\|_{L^\infty(0, T; L^2(\mathbb{T}^3))} \|\nabla_x u\|_{L^p((0, T) \times \Omega)} \leq \|u\|_{\mathcal{H}}^2,$$

which means that $(u^n \cdot \nabla_x) u^n$ is uniformly bounded in $L^{1+\eta}$ and thus it is uniformly integrable. By Vitali's convergence theorem

$$\int_0^T \int_{\mathbb{T}^3} (u^n \cdot \nabla_x) u^n \cdot \phi dx dv dt \rightarrow \int_0^T \int_{\mathbb{T}^3} (u \cdot \nabla_x) u \cdot \phi dx dv dt \quad (3.19)$$

for all divergence free smooth ϕ with compact support in $(0, T)$.

Similarly, up to a subsequence $\tau(Du^n) \rightarrow \tau(Du)$ a.e. and by (2.1) it is bounded in $L^{p'}(0, T; L^{p'}(\mathbb{T}^3))$. Vitali's convergence theorem implies that $\tau(Du^n) \rightarrow \tau(Du)$ strongly in $L^{p'-\eta}(0, T; L^{p'-\eta}(\mathbb{T}^3))$ for some $\eta > 0$. On the other hand, by Banach–Alaoglu Theorem the sequence $\{\tau(Du^n)\}_{n \in \mathbb{N}}$ converges weakly in $L^{p'}(0, T; L^{p'}(\mathbb{T}^3))$ to $\tau(Du)$ and by weak sequential lower semicontinuity of the norm $\tau(Du) \in L^{p'}(0, T; L^{p'}(\mathbb{T}^3))$. Whence

$$\int_0^T \int_{\mathbb{T}^3} \tau(Du^n) : D\phi dx dv dt \rightarrow \int_0^T \int_{\mathbb{T}^3} \tau(Du) : D\phi dx dv dt \quad (3.20)$$

for all divergence free smooth $\phi \in C^\infty$ with compact support in t . Convergence and boundedness of the external force follows by similar arguments. Altogether, (3.18)–(3.20) imply that for all divergence free smooth ϕ with compact support in the time variable we have

$$\int_0^T \int_{\mathbb{T}^3} (-u \cdot \partial_t \phi + (u \cdot \nabla_x) u \cdot \phi + \tau(Du) : D\phi) dx dv dt = -3 \int_0^T \int_{\mathbb{T}^3} \int_{\mathbb{R}^3} (u - v) \cdot \phi f dx dv dt. \quad (3.21)$$

By previously applied arguments we obtain that

$$u \in L^\infty(0, T; \dot{W}^{1,p}(\mathbb{T}^3)) \cap L^\infty(0, T; W^{1,2}(\mathbb{T}^3)) \cap L^2(0, T; W^{2,2}(\mathbb{T}^3))$$

and since $\partial_t u \in L^2(0, T; L^2(\mathbb{T}^3))$, we conclude that $u \in C([0, T]; L^2(\mathbb{T}^3))$ and thus $u \in \mathcal{H}$.

Finally, due to the sufficient regularity of u we may replace (3.21) by equation from point 4 of Definition 2.1 and by a density argument extend the class of admissible test functions to $W^{1,2}(\mathbb{T}^3) \cap \dot{W}_{div}^{1,p}(\mathbb{T}^3)$.

The proof of the fact that $f \in \mathcal{X}$ and that f satisfies point 3 of Definition 2.1 is straightforward, since f is at this point still regularized (i.e. $f = f_\epsilon \in C^2([0, T] \times \mathbb{T}^3 \times \mathbb{R}^3)$).

□

3.3 Local convergence with the regularized solutions

Until now we proved existence of solutions to the regularized system (3.1). The next goal is to converge with ϵ to 0 and to obtain local-in-time existence for (1.3).

Proposition 3.5. *Let $p \geq \frac{11}{5}$ and (f_ϵ, u_ϵ) be a solution to system (3.1) constructed as a limit of the approximate solutions as proved in Proposition 3.4. Then there exists $T^* \in (0, T]$, such that (f_ϵ, u_ϵ) satisfies the following estimates*

$$\|M_\alpha f_\epsilon\|_{L^\infty[0, T^*]} \leq C(T^*), \quad \text{for } 0 \leq \alpha \leq 2, \quad (3.22)$$

$$\|u_\epsilon\|_{L^\infty(0, T^*; L^2_{div}(\mathbb{T}^3)) \cap L^p(0, T^*; \dot{W}_{div}^{1,p}(\mathbb{T}^3))} \leq C(T^*), \quad (3.23)$$

$$\left\| \int_{\mathbb{R}^3} (\theta_\epsilon * u_\epsilon - v) \gamma_\epsilon f_\epsilon dv \right\|_{L^2(0, T^*; L^2(\mathbb{T}^3))} \leq C(T^*), \quad (3.24)$$

where $C(T^*)$ is a positive constant depending on the initial data and T^* .

Proof. We multiply equation (3.1)₁ by $|v|^2$ and integrate to obtain

$$\begin{aligned} 0 &= \frac{d}{dt} M_2 f_\epsilon + \int_{\mathbb{T}^3 \times \mathbb{R}^3} |v|^2 v \cdot \nabla_x f_\epsilon dx dv + \int_{\mathbb{T}^3 \times \mathbb{R}^3} |v|^2 \operatorname{div}_v [F_{CS}(f_\epsilon) f_\epsilon + \gamma_\epsilon (\theta_\epsilon * u_\epsilon - v) f_\epsilon] dx dv \\ &= \frac{d}{dt} M_2 f_\epsilon - 2 \int_{\mathbb{T}^3 \times \mathbb{R}^3} v \cdot F_{CS}(f_\epsilon) f_\epsilon dx dv - 2 \int_{\mathbb{T}^3 \times \mathbb{R}^3} \gamma_\epsilon v \cdot (\theta_\epsilon * u_\epsilon - v) f_\epsilon dx dv \end{aligned}$$

and since by substituting x with y and v with w (as in the estimate of S_1 in the proof of Proposition 3.5) we have

$$\int_{\mathbb{T}^3 \times \mathbb{R}^3} v \cdot F_{CS}(f_\epsilon) f_\epsilon dx dv \leq -C \int_{\mathbb{T}^3 \times \mathbb{R}^3} |w - v|^2 \psi(|x - y|) f_\epsilon(t, w, y) f_\epsilon(t, u, x) dy dw dx dv \leq 0,$$

we deduce the inequality

$$\frac{d}{dt} M_2 f_\epsilon \leq 2 \int_{\mathbb{T}^3 \times \mathbb{R}^3} \gamma_\epsilon v \cdot (\theta_\epsilon * u_\epsilon - v) f_\epsilon dx dv. \quad (3.25)$$

Next we test the weak formulation for u_ϵ by u_ϵ to get

$$\begin{aligned} \frac{1}{3} \frac{d}{dt} \|u_\epsilon\|_2^2 + \frac{2}{3} c_1 \kappa \|\nabla_x u_\epsilon\|_p^p &\leq -2 \int_{\mathbb{T}^3 \times \mathbb{R}^3} f_\epsilon (\theta_\epsilon u_\epsilon - v) \cdot u_\epsilon \gamma_\epsilon dx dv \\ &= 2 \int_{\mathbb{T}^3 \times \mathbb{R}^3} f_\epsilon (\theta_\epsilon * u_\epsilon - v) \cdot (\theta_\epsilon * u_\epsilon - u_\epsilon) \gamma_\epsilon dx dv - 2 \int_{\mathbb{T}^3 \times \mathbb{R}^3} f_\epsilon (\theta_\epsilon * u_\epsilon - v) \cdot \theta_\epsilon * u_\epsilon \gamma_\epsilon dx dv. \end{aligned} \quad (3.26)$$

We add (3.25) and (3.26) obtaining

$$\begin{aligned} \frac{d}{dt} \left(M_2 f_\epsilon + \frac{1}{3} \|u_\epsilon\|_2^2 \right) + \frac{2}{3} c_1 \kappa \|\nabla_x u_\epsilon\|_p^p + 2 \int_{\mathbb{T}^3 \times \mathbb{R}^3} \gamma_\epsilon |\theta_\epsilon * u_\epsilon - v|^2 f_\epsilon dx dv \\ \leq 2 \int_{\mathbb{T}^3 \times \mathbb{R}^3} \gamma_\epsilon (\theta_\epsilon * u_\epsilon - v) \cdot (\theta_\epsilon * u_\epsilon - u_\epsilon) f_\epsilon dx dv. \end{aligned} \quad (3.27)$$

Hölder's and Young's inequalities yield the following estimate of the right-hand side

$$\begin{aligned} 2 \int_{\mathbb{T}^3 \times \mathbb{R}^3} \gamma_\epsilon (\theta_\epsilon * u_\epsilon - v) \cdot (\theta_\epsilon * u_\epsilon - u_\epsilon) f_\epsilon dx dv \\ \leq \int_{\mathbb{T}^3 \times \mathbb{R}^3} \gamma_\epsilon |\theta_\epsilon * u_\epsilon - v|^2 f_\epsilon dx dv + C \|\theta_\epsilon * u_\epsilon - u_\epsilon\|_6^2 \|m_0 \sqrt{f_\epsilon}\|_3^2 \\ \leq \int_{\mathbb{T}^3 \times \mathbb{R}^3} \gamma_\epsilon |\theta_\epsilon * u_\epsilon - v|^2 f_\epsilon dx dv + \eta \|\theta_\epsilon * u_\epsilon - u_\epsilon\|_6^p + C(\eta) \|m_0 f_\epsilon\|_{\frac{3}{2}}^{\frac{p}{p-2}}. \end{aligned} \quad (3.28)$$

By Young's inequality for convolutions we have $\|\theta_\epsilon * u_\epsilon - u_\epsilon\|_6^p \leq 2^p \|\nabla_x u_\epsilon\|_p^p$, thus choosing a suitable η we obtain

$$\frac{d}{dt} \left(M_2 f_\epsilon + \frac{1}{3} \|u_\epsilon\|_2^2 \right) + \frac{1}{3} c_1 \kappa \|\nabla_x u_\epsilon\|_p^p + \int_{\mathbb{T}^3 \times \mathbb{R}^3} \gamma_\epsilon |\theta_\epsilon * u_\epsilon - v|^2 f_\epsilon dx dv \leq C \|m_0 f_\epsilon\|_{\frac{3}{2}}^{\frac{p}{p-2}}.$$

We apply Lemma A.1 and uniform L^1 and L^∞ bounds on f_ϵ given by (3.6) to get

$$\|m_0 f_\epsilon\|_{\frac{3}{2}} \leq C(M_{\frac{3}{2}} f_\epsilon)^{\frac{2}{3}} \leq C(M_2 f_\epsilon)^{\frac{1}{2}}$$

which yields

$$\frac{d}{dt}\left(M_2 f_\epsilon + \frac{1}{3}\|u_\epsilon\|_2^2\right) + \frac{1}{3}c_1\kappa\|\nabla_x u_\epsilon\|_p^p + \int_{\mathbb{T}^3 \times \mathbb{R}^3} \gamma_\epsilon |\theta_\epsilon * u_\epsilon - v|^2 f_\epsilon dx dv \leq C(M_2 f_\epsilon)^{\frac{p}{2p-4}}.$$

By the nonlinear version of Gronwall's lemma (Lemma A.3) there exists $T^* \in (0, T)$ (with $T^* = T$ for $p \geq 4$) such that for $t \in [0, T^*]$, we have

$$\left(M_2 f_\epsilon + \frac{1}{3}\|u_\epsilon\|_2^2\right)(t) + \frac{1}{3}c_1\kappa \int_0^t \|\nabla_x u_\epsilon\|_p^p ds + \int_0^t \int_{\mathbb{T}^3 \times \mathbb{R}^3} |\theta_\epsilon * u_\epsilon - v|^2 \gamma_\epsilon f_\epsilon dx dv ds \leq C(T^*). \quad (3.29)$$

This proves (3.22) and (3.23). It remains to prove (3.24). We do this by using (3.4): we estimate

$$G_\epsilon = \int_{\mathbb{R}^3} (\theta_\epsilon * u_\epsilon - v) \gamma_\epsilon f_\epsilon dv$$

in $L^2(0, T^*; L^2(\mathbb{T}^3))$ by a combination of terms that are bounded thanks to the energy estimate (3.29) and terms that appear on the left-hand side of (3.4); then we move the bad terms to the left hand side of (3.29) and finish the estimation. We perform the computations only for $p < 3$. For $p \geq 3$ slightly different argument is needed, but due to higher regularity of solutions, the proof is in fact easier. We have

$$\begin{aligned} \int_0^{T^*} \|G_\epsilon\|_2^2 dt &\leq C \left(\int_0^{T^*} \int_{\mathbb{T}^3} \left(\int_{\mathbb{R}^3} |\theta_\epsilon * u_\epsilon| \gamma_\epsilon f_\epsilon dv \right)^2 dx dt + \int_0^{T^*} \int_{\mathbb{T}^3} \left(\int_{\mathbb{R}^3} |v| \gamma_\epsilon f_\epsilon dv \right)^2 dx dt \right) \\ &=: A + B. \end{aligned}$$

First we estimate A :

$$A \leq \int_0^{T^*} \int_{\mathbb{T}^3} |\theta_\epsilon * u_\epsilon|^2 (m_0 f_\epsilon)^2 dx dt \leq \int_0^{T^*} \|u_\epsilon\|_\infty^2 dt \cdot \sup_{t \leq T^*} \int_{\mathbb{T}^3} (m_0 f_\epsilon)^2 dx. \quad (3.30)$$

Lemma A.1 implies that

$$\int_{\mathbb{T}^3} (m_0 f_\epsilon)^2 dx \leq \int_{\mathbb{T}^3} (m_0 f_\epsilon)^{\frac{5}{3}} dx \cdot \|m_0 f_\epsilon\|_\infty^{\frac{1}{3}} \leq CM_2 f_\epsilon \|f_\epsilon\|_\infty^{\frac{1}{3}} \stackrel{(3.6), (3.29)}{\leq} C(T^*) \mathcal{R}, \quad (3.31)$$

where \mathcal{R} denotes the radius of the support of f_ϵ in v . Moreover, the proof of Lemma 3.1 implies that

$$\mathcal{R} \leq Ce^{CT^*} \left(R + T^* \sqrt{M_2 f_\epsilon} + \int_0^{T^*} \|u_\epsilon\|_\infty dt \right), \quad (3.32)$$

thus combining (3.30) and (3.31) we get

$$A \leq C(T) \int_0^{T^*} \|u_\epsilon\|_\infty^2 dt \left(\sqrt{M_2 f_\epsilon} + \int_0^{T^*} \|u_\epsilon\|_\infty dt + 1 \right) \leq C(T) \left[\left(\int_0^{T^*} \|u_\epsilon\|_\infty^2 dt \right)^{\frac{3}{2}} + 1 \right]. \quad (3.33)$$

To estimate $\|u_\epsilon\|_\infty$, we use the Gagliardo–Nirenberg inequality obtaining

$$\begin{aligned} \|u_\epsilon\|_\infty &\leq C \left(\|\nabla_x u_\epsilon\|_{3p}^{\frac{3-p}{2}} + \|u_\epsilon\|_2^{\frac{3-p}{p}} \right) \|u_\epsilon\|_{\frac{3p}{3-p}}^{\frac{p-1}{2}} \\ &\leq C \left(\|\nabla_x u_\epsilon\|_{3p}^{\frac{3-p}{2}} + \|u_\epsilon\|_2^{\frac{3-p}{p}} \right) \left(\|\nabla_x u_\epsilon\|_p^{\frac{p-1}{2}} + \|u_\epsilon\|_2^{\frac{p-1}{2}} \right), \end{aligned}$$

which implies that (note that $\frac{p}{3-p} > 1$ for $p > \frac{3}{2}$)

$$\begin{aligned} & \int_0^{T^*} \left(\|\nabla_x u_\epsilon\|_{3p}^{\frac{3-p}{2}} + \|u_\epsilon\|_2^{\frac{3-p}{p}} \right) \left(\|\nabla_x u_\epsilon\|_p^{\frac{p-1}{2}} + \|u_\epsilon\|_2^{\frac{p-1}{2}} \right) dt \\ & \leq^{H(\frac{p}{3-p})} C \left(\int_0^{T^*} \|\nabla_x u_\epsilon\|_{3p}^p dt \right)^{\frac{3-p}{p}} \left(\int_0^{T^*} \|\nabla_x u_\epsilon\|_p^{\frac{p-1}{2p-3}} dt \right)^{\frac{2p-3}{p}} + l.o.t., \end{aligned}$$

where l.o.t. denotes lower order terms connected with the presence of $\|u_\epsilon\|_2$ which is bounded on $(0, T^*)$. Therefore

$$A \leq C(T) \left(\int_0^{T^*} \|\nabla_x u_\epsilon\|_{3p}^p dt \right)^{\frac{9-3p}{2p}} \left(\int_0^{T^*} \|\nabla_x u_\epsilon\|_p^{\frac{p-1}{2p-3}} dt \right)^{\frac{6p-9}{2p}} + l.o.t.$$

We use the fact that $\frac{p-1}{2p-3} \leq 1$ for $p \geq 2$ and Young's inequality with exponent $\frac{2p}{9-3p}$ (which is greater than 1 for $p > \frac{9}{5}$) to get for arbitrary $\eta > 0$

$$A \leq \eta \int_0^{T^*} \|\nabla_x u_\epsilon\|_{3p}^p dt + C(\eta) \left(\int_0^{T^*} \|\nabla_x u_\epsilon\|_p^p dt \right)^{\frac{6p-9}{5p-9}} + C. \quad (3.34)$$

On the other hand, for B we have by Lemma A.1

$$B \leq \int_0^{T^*} \int_{\mathbb{T}^3} (m_1 f_\epsilon)^{\frac{5}{4}} \cdot (m_1 f_\epsilon)^{\frac{3}{4}} \leq TM_2 f_\epsilon \|m_1 f_\epsilon\|_\infty^{\frac{3}{4}} \stackrel{(3.6), (3.29)}{\leq} C(T) \mathcal{R}^3.$$

We estimate \mathcal{R} again using (3.32) obtaining

$$B \leq C(T) \left[\left(\int_0^{T^*} \|u_\epsilon\|_\infty dt \right)^3 + 1 \right] \leq C(T) \left[\left(\int_0^{T^*} \|u_\epsilon\|_\infty^2 dt \right)^{\frac{3}{2}} + 1 \right]$$

and this is the estimate with exactly the same right-hand side as (3.33). Thus from this point we proceed like in the estimation of A altogether getting

$$\int_0^{T^*} \|G_\epsilon\|_2^2 dt \leq A + B \leq \eta \int_0^{T^*} \|\nabla_x u_\epsilon\|_{3p}^p dt + C(\eta) \left[\left(\int_0^{T^*} \|\nabla_x u_\epsilon\|_p^p dt \right)^{\frac{6p-9}{5p-9}} + 1 \right] \quad (3.35)$$

for all $\eta > 0$ and note that due to energy estimate (3.29) the second summand on the right-hand side is bounded on $[0, T^*]$. We apply this estimate to (3.4) which after taking a sufficiently small η enables us to move the term with $\|u_\epsilon\|_{3p}^p$ to the left-hand side which results in

$$\sup_{t \leq T^*} \|u_\epsilon\|_{W^{1,2}(\mathbb{T}^3)}^2 + \|\nabla_x^{(2)} u_\epsilon\|_{L^2(0, T^*; L^2(\mathbb{T}^3))}^2 + \frac{1}{2} \|\nabla_x u_\epsilon\|_{L^p(0, T^*; L^{3p}(\mathbb{T}^3))}^p \leq C, \quad (3.36)$$

where C depends on $\|u_{0,\epsilon}\|_2^2, M_2 f_{0,\epsilon}, R, \|f_\epsilon\|_\infty, T^*, T, p$, all of which are either fixed or bounded independently of ϵ . Finally applying (3.36) to (3.35) finishes the proof of (3.24). \square

The following corollary combines all the necessary local in time uniform estimates of u_ϵ and f_ϵ proved throughout this section.

Corollary 3.1. *The solution (f_ϵ, u_ϵ) satisfies Propositions 3.3 and 3.5 uniformly with respect to $\epsilon > 0$ on the time interval $[0, T^*]$.*

Proof. To prove estimate (i) we notice that by Proposition 3.2 the $\|\cdot\|_{\mathcal{H}}$ norm of u_n^ϵ depends only on $\|u_0\|_{W^{1,2}(\mathbb{T}^3)}$, which is fixed and on $\|G\|_{L^2(0,T;L^2(\mathbb{T}^3))}$ which by Proposition 3.5 is uniformly bounded with respect to ϵ on the time interval $[0, T^*]$. Therefore also $\|u^\epsilon\|_{\mathcal{H}}$ is uniformly bounded on $[0, T^*]$. The exactly same argument is valid for the ϵ -independent estimate (ii). Estimate (iii) was already proved above, while estimates (iv) and (v) were shown to hold with constants depending on the constants from (now proved to be ϵ independent) estimates (i) – (iii) and the estimate of the support of f_ϵ^n from Proposition 3.3. Therefore (iv) and (v) hold with ϵ - and T^* -independent constants on the time interval $[0, T^*]$ if only the support of f_ϵ is uniformly bounded.

It remains to show that f_ϵ satisfies (3.8) with \mathcal{R} independent of ϵ . By Lemmas 3.1 and 3.2 each iterative solution f_ϵ^n has a support contained in a ball of radius \mathcal{R}_ϵ with $\mathcal{R}_\epsilon(t)$ depending on $\|u_\epsilon^n\|_{L^2(0,T;W^{2,2}(\mathbb{T}^3))}$ and $\|M_1 f_\epsilon^n\|_\infty$ (and R which depends only on the initial data). However, by Proposition 3.5, these quantities are uniformly bounded on $[0, T^*]$ thus so is \mathcal{R}_ϵ . Finally f_ϵ inherits the uniform boundedness of the support from f_ϵ^n as an L^∞ -limit. \square

Proof of Theorem 2.1 – local existence. With the uniform bounds from Corollary 3.1, it remains to let $\epsilon \rightarrow 0$. By virtue of those uniform bounds it follows that u_ϵ is uniformly bounded in $\mathcal{H} \hookrightarrow L^2(0, T; W^{2,2}(\mathbb{T}^3))$ and $\partial_t u^n$ is uniformly bounded in $L^2(0, T; L^2(\mathbb{T}^3))$. Since it holds $W^{2,2}(\mathbb{T}^3) \hookrightarrow W^{1,2}(\mathbb{T}^3) \hookrightarrow L^2(\mathbb{T}^3)$, by the Aubin–Lions lemma, we may extract from u_ϵ a strongly convergent subsequence in $L^2(0, T^*; W^{1,2}(\mathbb{T}^3))$. Then we pass to the limit in every term in Definition 2.1 (iv) (similarly to the proof of Proposition 3.4) obtaining the weak formulation for u . The convergence of f_ϵ is performed in the same way as in [2]. To prove that $u \in C(0, T; L^2_{div}(\mathbb{T}^3))$ and $f \in C(0, T; L^2(\mathbb{T}^3 \times \mathbb{R}^3))$, we use estimates (i) and (ii) from Corollary 3.1 and the well-know result on the Gelfand triplet. This finishes the proof of local existence of solutions in the sense of Definition 2.1. \square

3.4 Step 6: Global existence

This part is dedicated to show that $T^* = T$. The previous subsection gave the local existence for the original problem. To show the existence on the whole time interval $[0, T]$ is sufficient to construct a priori estimate controlling traces in time which allow to prolong the solution till time T . As $\theta_\epsilon u_\epsilon \rightarrow u_\epsilon$ in the regularity class, where u_ϵ is bounded, it is not difficult to show that estimate (3.27) takes the form

$$\left(M_2 f + \frac{1}{3}\|u\|_2^2\right)(t) + \frac{2}{3}c_1\kappa \int_0^t \|\nabla_x u\|_p^p ds + \int_0^t \int_{\mathbb{T}^3 \times \mathbb{R}^3} |u - v|^2 f dx dv ds \leq \left(M_2 f + \frac{1}{3}\|u\|_2^2\right)(0).$$

Note that the right-hand side depends only on the initial data of our problem. We were not able to use this argument previously and this led to the necessity to prove the local existence result first.

Then in order to apply Proposition 3.1 we estimate the drag force

$$G = \int_{\mathbb{R}^3} (u - v) f dv$$

in the same way as we estimated G_ϵ in the proof of Proposition 3.5. So we find the better integrability of u . Hence for all $t \in [0, T^*)$ we are able to construct the a priori estimate

without dependence from T^* , guaranteeing that, by continuity $u(T^*) \in W^{2,2}(\mathbb{T}^3)$, $f(T^*) \in L^1 \cap L^\infty(\mathbb{T}^3 \times \mathbb{R}^3)$ and by Proposition 3.3 point (iii), $\nabla_v f(T^*) \in L^3(\mathbb{T}^3 \times \mathbb{R}^3)$. Hence we are able to prolong the solution over T^* . We proved $T^* = T$, hence the solution exists in fact in the whole time interval $(0, T)$.

4 Uniqueness of solutions

Let us denote $\omega := u^1 - u^2$, $g := f^1 - f^2$, where (f_1, u_1) and (f_2, u_2) are two supposedly different solutions to (1.3) subjected to the same initial data (f_0, u_0) . First we deal with the fluid part of the problem. We subtract weak formulation of u_2 from the weak formulation of u_1 obtaining the weak formulation for ω , which we test by $\psi = \omega$ (it is an admissible test function due to Definition 2.1), obtaining

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\omega\|_2^2 + c_1 \kappa \|\nabla_x \omega\|_2^2 &\leq 3 \int_{\mathbb{T}^3 \times \mathbb{R}^3} |\omega|^2 f^1 dx dv + 3 \int_{\mathbb{T}^3 \times \mathbb{R}^3} \omega |u^2 - v| f^1 - f^2 dx dv \\ &\quad + \int_{\mathbb{T}^3} |\omega|^2 |\nabla_x u^1| dx =: I_1 + II_1 + III_1. \end{aligned} \quad (4.1)$$

We estimate I_1 , II_1 and III_1 separately. By virtue of Lemma A.1, we have

$$I_1 = \int_{\mathbb{T}^3} |\omega|^2 m_0 f^1 dx \leq \int_{\mathbb{T}^3} |\omega|^2 (m_3 f^1)^{\frac{1}{2}} dx \leq \|\omega\|_4^2 (M_3 f^1)^{\frac{1}{2}}.$$

Compactness of the support of f in v implies that $M_3 f^1 \leq C$, from which it follows that

$$I_1 \leq C \|\omega\|_2^{\frac{1}{2}} \|\omega\|_6^{\frac{3}{2}} \leq C (\|\omega\|_2^{\frac{1}{2}} \|\nabla_x \omega\|_2^{\frac{3}{2}} + \|\omega\|_2^2) \stackrel{Y(4)}{\leq} C(\delta) \|\omega\|_2^2 + \delta \|\nabla_x \omega\|_2^2. \quad (4.2)$$

Furthermore,

$$\begin{aligned} II_1 &\leq \int_{\mathbb{T}^3 \times \mathbb{R}^3} |\omega| |u^2| m_0 |g| dx dv + \int_{\mathbb{T}^3 \times \mathbb{R}^3} |\omega| m_1 |g| dx dv \\ &\stackrel{Y(2)}{\leq} \frac{1}{2} (\|u^2\|_\infty^2 + 1) \|\omega\|_2^2 + \frac{1}{2} \|m_0 |g|\|_2^2 + \frac{1}{2} \|m_1 |g|\|_2^2 \end{aligned}$$

and as due to Definition 2.1 (i) $\text{supp } f(x, \cdot, t) \subset B(\mathcal{R}(T))$,

$$\|m_\alpha |g|\|_2^2 = \int_{\mathbb{T}^3} \left(\int_{\mathbb{R}^3} |v|^\alpha |g| dv \right)^2 dx \leq C \mathcal{R}(T)^{2\alpha+6} \|g\|_2^2,$$

which implies that

$$II_1 \leq \frac{1}{2} (\|u^2\|_\infty^2 + 1) \|\omega\|_2^2 + C \|g\|_2^2. \quad (4.3)$$

Finally we estimate III_1 :

$$III_1 \stackrel{H(2)}{\leq} \|\omega\|_4^2 \|\nabla_x u^1\|_2 \leq C (\|\omega\|_2^{\frac{1}{2}} \|\nabla_x \omega\|_2^{\frac{3}{2}} + \|\omega\|_2^2) \stackrel{Y(4)}{\leq} C(\delta) \|\omega\|_2^2 + \delta \|\nabla_x \omega\|_2^2, \quad (4.4)$$

where we used the fact that $u^1 \in L^\infty(0, T; W^{1,2}(\mathbb{T}^3))$ to control $\|\nabla_x u^1\|_2$. Inequality (4.1) and estimates (4.2)–(4.4) imply that

$$\frac{1}{2} \frac{d}{dt} \|\omega\|_2^2 + \frac{1}{2} c_1 \kappa \|\nabla_x \omega\|_2^2 \leq C (\|u^2\|_\infty^2 + 1) \|\omega\|_2^2 + C \|g\|_2^2. \quad (4.5)$$

This finishes the estimates of the fluid part of the system.

Next we estimate the particle part in $L^\infty(0, T; L^2(\mathbb{T}^3 \times \mathbb{R}^3))$. We subtract the weak formulation for f^2 from one for f^1 obtaining the formulation for g which is tested by g ³

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|g\|_2^2 &= \int_{\mathbb{T}^3 \times \mathbb{R}^3} \left[F_{CS}(g)f^2 + F_{CS}(f^1)g + \omega f^1 + (u^2 - v)g \right] \cdot \nabla_v g dx dv \\ &=: I_2 + II_2 + III_2 + IV_2. \end{aligned} \quad (4.6)$$

Next we estimate I_2, II_2, III_2, IV_2 . We have

$$\begin{aligned} I_2 &= \int_{\mathbb{T}^3 \times \mathbb{R}^3} F_{CS}(g)f^2 \cdot \nabla_v g dx dv = - \int_{\mathbb{T}^3 \times \mathbb{R}^3} \operatorname{div}_v F_{CS}(g)f^2 g dx dv - \int_{\mathbb{T}^3 \times \mathbb{R}^3} F_{CS}(g)\nabla_v f^2 g dx dv \\ &=: I_{21} + I_{22} \end{aligned}$$

for

$$\begin{aligned} |I_{21}| &\stackrel{H(2)}{\leq} \|f^2\|_2 \|\operatorname{div}_v F_{CS}(g)\|_\infty \|g\|_2 \leq C \|g\|_2^2, \\ |I_{22}| &\stackrel{H(2)}{\leq} \|\nabla_v f^2\|_2 \|F_{CS}(g)\|_\infty \|g\|_2 \stackrel{(2.10)}{\leq} C \|g\|_2^2, \end{aligned}$$

where we use that g has a compact support in v . Furthermore,

$$II_2 = \int_{\mathbb{T}^3 \times \mathbb{R}^3} F_{CS}(f^1) \cdot \nabla_v |g|^2 dx dv = - \int_{\mathbb{T}^3 \times \mathbb{R}^3} \operatorname{div}_v F_{CS}(f^1) |g|^2 dx dv,$$

thus by (2.9) $|II_2| \leq C \|g\|_2^2$. Moreover

$$III_2 = \int_{\mathbb{T}^3 \times \mathbb{R}^3} f^1 \omega \cdot \nabla_v g dx dv = - \int_{\mathbb{T}^3 \times \mathbb{R}^3} \omega \cdot \nabla_v f^1 g dx dv,$$

thus

$$\begin{aligned} |III_2| &\stackrel{H(2)}{\leq} \int_{\mathbb{T}^3} \left(\int_{\mathbb{R}^3} |\nabla_v f^1|^2 dv \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^3} |g|^2 dv \right)^{\frac{1}{2}} \omega dx \\ &\leq \left(\int_{\mathbb{T}^3} \left(\int_{\mathbb{R}^3} |\nabla_v f^1|^2 dv \right)^{\frac{3}{2}} dx \right)^{\frac{1}{3}} \left(\int_{\mathbb{T}^3 \times \mathbb{R}^3} |g|^2 dx dv \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^3} |\omega|^6 dx \right)^{\frac{1}{6}} \leq \delta \|\nabla_x \omega\|_2^2 + \|\omega\|_2^2 + C(\delta) \|g\|_2^2, \end{aligned}$$

using again that the support of f is compact in the v -variable. For IV_2 , we integrate by parts obtaining

$$IV_2 = \int_{\mathbb{T}^3 \times \mathbb{R}^3} (u^2 - v) \cdot \nabla_v (g^2) dx dv = 3 \int_{\mathbb{T}^3 \times \mathbb{R}^3} |g^2| dx dv.$$

Finally, we combine (4.5), (4.6) with the estimates of $I_2 - IV_2$ to obtain

$$\frac{1}{2} \frac{d}{dt} (\|\omega\|_2^2 + \|g\|_2^2) + \frac{1}{4} c_1 \kappa \|\nabla_x \omega\|_2^2 \leq C(\|u^2\|_\infty^2 + 1) \|\omega\|_2^2 + C \|g\|_2^2.$$

We aim to use Gronwall's lemma to conclude that $\omega = 0$ and $g = 0$, which means that the solution is unique. Since $u^2 \in \mathcal{H}$ and $L^2(0, T; W^{2,2}(\mathbb{T}^3)) \hookrightarrow L^2(0, T; L^\infty(\mathbb{T}^3))$, it implies the integrability of $t \mapsto \|u^2\|_\infty^2$. Theorem 2.1 is proved.

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³Let us note that due to the effort we put into estimates of $\nabla_v f$ and $\nabla_x f$ we made sure that f is a good test function in point 3 of Definition 2.1.

A Appendix A

We present the basic tools used throughout the paper.

We present two crucial lemmas from [4].

Lemma A.1. *Let $\beta > 0$ and g be a nonnegative function in $L^\infty([0, T] \times \mathbb{T}^3 \times \mathbb{R}^3)$. The following estimate holds for any $\alpha < \beta$:*

$$m_\alpha g(t, x) \leq \left(\frac{4}{3} \pi \|g(t, x, \cdot)\|_\infty + 1 \right) m_\beta g(t, x)^{\frac{\alpha+3}{\beta+3}},$$

for a.a. (t, x) .

Proof. The proof can be found in [4], page 9 (Lemma 1). \square

Lemma A.2. *For $T > 0$, let $\{a_n\}$ be a sequence of nonnegative continuous functions defined on $[0, T]$ satisfying the relation:*

$$a_{n+1}(t) \leq A + B \int_0^t a_n(s) ds + C \int_0^t a_{n+1}(s) ds, 0 \leq t \leq T,$$

where A, B and C are nonnegative constants, Then there exists a positive constant K such that for all $n \in \mathbb{N}$

$$a_n(t) \leq K e^{Kt}.$$

Proof. The proof can be found in [4], page 15 (Lemma 3). \square

We include the formulation of the classical Gronwall's lemma with it's less popular non-linear varieties.

Lemma A.3 (Gronwall's lemma). *Let f be a nonnegative function satisfying inequality*

$$f(t) \leq c + \int_{t_0}^t (a(s)f(s) + b(s)f^q(s)) ds, \quad c \geq 0, \quad q > 1,$$

where a and b are nonnegative, integrable functions for $t \geq t_0$. Then we have

$$f(t) \leq c \left[e^{(1-q) \int_{t_0}^t a(s) ds} - c^{-1} (q-1) \int_{t_0}^t b(s) e^{(1-q) \int_s^t a(r) dr} ds \right]^{\frac{1}{q-1}}$$

for $t \in [t_0, h]$ for $h > 0$ provided that

$$c < \left[e^{(1-q) \int_{t_0}^{t_0+h} a(s) ds} \right]^{\frac{1}{q-1}} \left[(q-1) \int_{t_0}^{t_0+h} b(s) ds \right]^{\frac{1}{1-q}}.$$

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